Int. J. Heat Mass Transfer. Vol. 3, pp. 188–197. Pergamon Press 1961. Printed in Great Britain.

ON AERODYNAMICS OF RAREFIED GASES AND PROBLEMS OF HEAT TRANSFER

A. S. PREDVODITELEV
Department of Molecular Physics, Moscow State University, U.S.S.R.

(*Received* 23 *January* 1961)

Abstract—Poor agreement between experimental data and theoretical expressions for acoustic dispersion in a rarefied gas which are obtained on the basis of the Burnett and super-Burnett solutions of the Boltzmann equation is shown in the present paper; the classical theory based on the Navier–Stokes hydrodynamics brings more successful results. The above-mentioned appears to be valid also when considering a temperature jump and slip. Contradictions between the Euler equations and the molecular-kinetic theory are also revealed while considering the conformability of these equations to the rarefied gas. The generalization of the Euler equations is fulfilled in the paper by introducing a parameter of non-ideal continuity.

Résumé—Cet article montre le mauvais accord qui existe entre les données expérimentales et les expressions théoriques obtenues pour la dispersion acoustique dans les gaz raréfiés d'après les solutions Burnett et super-Burnett de l'équation de Boltzmann; la théorie classique basée sur l'hydrodynamique de Navier–Stokes apporte des résultats plus heureux. La conclusion ci-dessus semble également être valable quand on considère un saut de température et un glissement. Des contradictions apparaissent également entre les équations d'Euler et la théorie cinétique moléculaire quand on applique ces équations aux gaz raréfiés. La généralisation des équations d'Euler est faite dans cet article en introduisant un paramètre exprimant que la continuité n'est pas parfaite.

Zusammenfassung—Für die akustische Dispersion in verdünnten Gasen wird die geringe Übereinstimmung zwischen den experimentellen und theoretischen Werten, wie sie sich auf Grund der Burnett- und Super-Burnett-Lösung der Boltzmanngleichung ergeben, aufgezeigt. Die klassischen Navier–Stokessche Theorie vermittelt bessere Ergebnisse. Das scheint auch für Temperatursprünge zu gelten. Widersprüche zwischen den Eulerschen Gleichungen und der molekular-kinetischen Theorie ergaben sich bei dem Versuch, die Gleichungen auf die verdünnten Gase anzuwenden. Die Verallgemeinerung der Eulerschen Gleichungen wurde hier dadurch erreicht, dass man einen Parameter nicht-idealer Kontinuität einführte.

Аннотация—В статье показывается, что теоретические выражения для акустической дисперсии в разреженном газе, полученные на основе барнетовских и супербарнетовских решений уравнения Больумана, плохо согласуются с данными экспериментов; более удачные результаты даёт классическая теория, основанная на гидродинамике Навье-Стокса. Это оказывается справедливым и для случая рассмотрения явлений температурного скачка и скольжения.

При рассмотрении уравнений Эйлера применительно к разреженному газу также вскрываются их противоречия с молекулярно-кинетической теорией. В работе проведено обобщение уравнпний Эйлера путём введения параметра неидеальной сплошности.

NOMENCLATURE

p, pressure;
ω, circular frequency;
η, dynamic viscosity;
T_w, wall temperature;
T_g, gas temperature;
k, temperature jump coefficient;
h, mean free path of molecules;
f, measure of affinity with molecules of a solid body;
W, flow velocity;
U, slip velocity near a wall;
R, gas constant;
ζ, slip coefficient;

M, molecular weight of a gas; Mach local number;
u, v, w, components of molecular velocity;
F, function of molecular distribution;
C, shortest distance between rectilinear trajectories;
V, relative velocity of colliding molecules;
φ, angle;
$d\omega = du \, . \, dv \, . \, dw$;
ρ, density;
γ, ratio of specific heat;

$$\left.\begin{array}{ll} r, & \text{distance along radius} \\ l, & \text{distance along axis} \end{array}\right\} \text{polar system reference;}$$

θ, angle;
η_3, parameter of non-ideal continuity.

1

At present various problems of heat and mass transfer in rarefied gases represent a whole field of knowledge. There are scores of works in this direction. Designers of various reactive machines, engineers and scientists working on industrial processes involving low gas densities come across various problems on aerodynamics and heat and mass transfer theory in rarefied gases. Improving the construction of powerful gas pumps brings out the necessity for a profound understanding of the phenomena in rarefied gases. However, in spite of the intensive development of the molecular-kinetic theory the problems mentioned are still being solved by methods which raise serious doubts because of the contradictions they give. Thus, it follows that it is necessary to look through all the principal concepts which are the basis of the aerodynamic equations and laws of heat and mass transfer in rarefied gases.

Thus, first it is necessary to search for some methods of experimental examination of the problems mentioned.

2

A very direct and simple method for such an examination is based on the theory of propagation of acoustic waves in rarefied gases. Indeed, until the length of an acoustic wave exceeds by many times the average length of the free path of molecules, the acoustic wave will propagate normally provided that there are no substance transformations in the gas. But if the length of an acoustic wave is comparable with the average length of the free path of molecules, then in this case the phenomenon of acoustic dispersion begins. The laws of this dispersion may be predicted starting from various forms of equations of aerodynamics of a rarefied gas. Thus, there is a possibility of direct examination of the principal states of the equations indicated.

As is known the momentum equations as well as that of energy transfer are deduced in the modern molecular-kinetic theory on the basis of the solutions of the so-called Boltzmann integro-differential equation. The solution of the Boltzmann equation in its first approximation, i.e. when the velocity and temperature gradients effects on the average length of the free path of molecules can be neglected, brings to the gas motion equations of the Navier–Stokes form. The second approximation obtained by Burnett using the Enskog–Chapman method introduces into the system of momentum and heat flow equations essentially new terms which alter dispersion laws of acoustic waves. In this case, to some extent, the variation of velocity and temperature gradients on the average length of the free path of molecules has already been taken into account. The solution of the Boltzmann equation in its third approximation exists already. It is known as the super-Burnett solution. The solution gives new corrections for the equations of momentum and heat flow. Consequently, it must give a more precise expression for acoustic disperson in rarefied gases.

The dimensionless value $r = (p/\omega\eta)$ is used as the independent variable for the experimental examination of theoretical formulae of acoustic dispersion which are deduced from different aerodynamic equations of rarefied gases.

It is easy to show that this value is proportional to the ratio of the acoustic wave length to the average length of the free path of molecules. Consequently, it is convenient for the estimation of the emergence of acoustic dispersion.

The classical theory based on the Navier–Stokes equations gives as a result the known formulae of the Stokes–Kirchoff acoustic dispersion. For values of r exceeding 10, i.e. when we deal with relatively low acoustic frequencies and high pressures, the relative value of the sound absorption coefficient is not great. Therefore, the velocity of its propagation remains practically a constant value. Consequently, there is no acoustic dispersion.

The relative absorption of sound increases at decrease of r; the velocity of its propagation becomes the function of r and the Stokes–Kirchoff acoustic dispersion begins. The absorption is maximum if the value of r is equal to unity. The absorption is somewhat lower with further decrease of this value.

The corresponding calculations based on the Burnett and super-Burnett momentum equations give qualitatively the same results. At great values of r the coefficient of sound absorption as the function of r coincides with the corresponding functions obtained from the Burnett and super-Burnett equations. However, the divergence between the corresponding functions begins to increase with decrease of r. The highest divergence is at the value of $r = 1$. This divergence is so considerable that it can be easily found experimentally. In 1953 Greenspan and Thompson carried out a detailed experimental investigation of the velocity and absorption of ultrasound waves in rarefied gases. The results of these highly precise experiments revealed that the classical theory based on the Navier–Stokes hydrodynamics gives correct results in a considerably wider range of r than the theories based on the Burnett and super-Burnett equations. Moreover, the super-Burnett equations obtained from the more precise solutions of the Boltzmann equation gave the worst results.

3

However, the phenomenon of acoustic dispersion is not the only method of verification of concepts of aerodynamics of rarefied gases.

At Moscow State University, in the Department of Molecular Physics, the phenomena of slip of the gas flow and the temperature jump between a wall and a gas were investigated.

The theory of the temperature jump and flow slip near a wall, as is known, is based on the solution of the Boltzmann equation in its first approximation and was verified by Kundt, Warburg, Smolukhovskii, Lazarev, Millikan, Timiryazev and others.

If we designate the corresponding temperature of a wall and a gas as T_w and T_g, then the heat transfer between a wall and a gas is regulated by the formula:

$$k\frac{\mathrm{d}\theta}{\mathrm{d}n} = T_w - T_g$$

moreover, the temperature jump coefficient k according to molecular-kinetic calculations is equal to:

$$k = \frac{15}{4\pi}L\frac{2-f}{f}.$$

Here L is the average length of the free path of molecules and f is the measure of affinity with those of a solid body.

The analogical formula also exists for the slip of a rarefied gas flow near a wall. Let us designate by W the velocity of the flow; by U the slip velocity near a wall, then we get:

$$\zeta\frac{\mathrm{d}W}{\mathrm{d}n} = -U$$

and moreover, the slip coefficient ζ according to the molecular-kinetic calculations, has the form:

$$\zeta = \frac{2-f}{f}\frac{\eta}{p}\sqrt{\left(\frac{\pi RT}{2M}\right)} = \frac{A_0}{p}.$$

R is the gas constant, M is the molecular weight of the gas. Now from the above-mentioned relations for the slip constant A_0 and for the temperature jump coefficient k, we can get the following important formula:

$$A_0 = \frac{8}{15}kp.$$

As we have noted earlier all these formulae are valid only at extremely low values of both temperature gradient and velocity gradient along the average length of the free path of molecules. If the given gradients are high, then we should search after correct terms for the given formulae. It is quite possible to find these correct terms theoretically, but they can be determined experimentally as well.

Gribkova and Shtemenko developed a very

interesting method of simultaneous measurements of the temperature jump and slip under conditions of such flows of rarefied gas where it was quite possible to have considerable gradients of temperature and velocities along the average length of the free path of molecules. The method is based on longitudinal flow along a thin heated thread, i.e. the flow of a rarefied gas in a cylindrical tube in which the diameter of an internal continuous cylinder was very small in comparison with the external one. Gribkova developed the theory of such an apparatus which was used in the solution of this problem.

The results of the investigations carried out appeared to be quite unexpected. In spite of the fact that both the gradients of temperature and velocities along the average length of the free path of molecules were considerable, results of the application of the classical theory to the temperature jump and slip appeared to be correct.

After a thorough analysis of the results of their experiments the authors of these investigations came to the following conclusion. Apparently, at motion of a rarefied gas near the surface of a solid in cases when the range of temperature change along the average length of the free path of molecules is 1–10°C, and the velocity of the mass motion variation is 50–60 m/s near the surface of a body, the law of distribution of heat velocities corresponding to the solution of the Boltzmann equation (in the first approximation) occurs.

4

(1) The described disagreement of the molecular-kinetic theory with the observed facts makes us think seriously of the reasons which cause these contradictions.

If we assume that the solution methods of the Boltzmann equation, which were worked out by mathematicians, are quite correct, then we should find an explanation of the described contradictions in imperfections of this equation.

As is known, two operations of the function of molecular distribution F, according to heat velocities, are equated with each other in the Boltzmann equation.

The variation of the function F per unit in time, co-ordinates and velocities is calculated with the help of the first operation. It may be written in the following way:

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} + X\frac{\partial F}{\partial u} + Y\frac{\partial F}{\partial v} + Z\frac{\partial F}{\partial w} \qquad (1)$$

and moreover, we take into consideration the equalities:

$$\frac{\partial u}{\partial t} = X; \quad \frac{\partial v}{\partial t} = Y; \quad \frac{\partial w}{\partial t} = Z.$$

The variation of the same function per time unit due to collisions of molecules is calculated with the help of the second operation.

If components of the molecular velocity of the first type molecules are designated by u_1, v_1, w_1 and components of the second type molecules are designated by u_2, v_2, w_2 till the moment of their collision among themselves, then the measure of probability of their approach may be taken as equal to F_1F_2. The subscripts denote in this expression from which components of heat velocities we should take the function of distribution.

After collision, the components of velocities of molecules of both types acquire the following values:

$$(u_1', v_1', w_1'); \quad (u_2', v_2', w_2').$$

The product of $F_1'\,F_2'$ will serve as a measure of probability of their repulsion.

Thus, each collision of one molecular group with that of another will be characterized by a difference:

$$F_1'F_2' - F_1F_2 = \delta F_1F_2.$$

If we summarize all these differences along the whole phase volume, we shall get the variation of the function F per time unit which is due to collision of molecules,

$$\frac{DF}{Dt} = \int \ldots \int \delta F_1F_2 V \mathrm{d}b\mathrm{d}\omega_1\mathrm{d}\varphi \qquad (2)$$

Here the product $V\mathrm{d}b\mathrm{d}\omega_1\mathrm{d}\varphi$ denotes an element of a phase space; V denotes the relative velocity of colliding molecules, b is the shortest distance between rectilinear trajectories of molecules; φ is the angle between the plane of the trajectories of the relative motion of the first type molecule and a certain constant plane passing through V; finally, $d\omega$, denotes the product du, dv, dw.

Consider whether these two operations are always equal to each other. The first operation by its nature can give continuous values from the co-ordinates of a phase space. As for the second operation it is not evident. We are inclined to suppose that the second operation must bring discontinuous values because the variation δF_1F_2 is not always an infinitely small value. It means that we cannot equate in the main the operations with each other. As is known, both operations are equated without any proof that the variation δF_1F_2 is the continuous co-ordinate function of a phase space. In our opinion such equalization will be more valid, the higher the density of a gas. The validity of the given equalization becomes less lawful as a gas is rarefied. Therefore, the solutions of the Boltzmann equation for rarefied gas must give unsatisfactory results. Probably the reason lies in the fact that the phenomena of acoustic dispersion in rarefied gases and phenomena of the temperature jump and slip of the gas flow near a wall cannot be restricted by the solutions of the Boltzmann equation according to the Enskog–Chapman method.

(2) However, recently another possible point of view has been discussed in literature. We mean the works by Truesdell.

In one of his works he says the following:

"Up to now there is no solution of the Boltzmann equation. It is true that there is a great number of so-called solutions. But the Maxwell theory was a paradise for conjectures and dogmatism, and it was a desert for precise mathematics. The methods named in this theory as integration are without any exception purely formal. Their convergence has never been proved but moreover there are no indications that classical methods of solution have significance for this equation. The above mentioned refers in equal degree to investigations of physicists as well as to the outstanding work by the great mathematician Gilbert." And further: "All the classical investigations are extremely overloaded with calculations. A great number of investigations are devoted to the study of the Boltzmann equation and to attempts to solve it with the help of special series. One always speaks of approximation and the order of magnitude, etc. However, complexity of the analysis makes difficult the understanding of the principal course of arguments."

In his work Truesdell goes further as he has doubts about the concept of the gas-kinetic theory and speaks of a modern crisis in the kinetic theory of gases. In his work, under such a title, he analyses the already formed situation in the kinetic theory of gases and shows that the problem of convergence of successive approximations is not trivial at all. In one concrete example he illustrates that there might exist such cases where all the approximations appeared to be worse than the first one, which is the asymptotic solution. It is quite possible that in a strict statement of the problem this asymptotic solution will be closer to the Navier–Stokes equations than all the existing approximate solutions of the Boltzmann equation. Now it is very difficult to say which point of view mentioned above is correct. It is clear that modern techniques are necessary to begin to solve this problem which is of great importance and deserves the most serious attention on the part of physicists and mathematicians.

5

(1) The well-known imperfection of our concepts in the kinetic theory may be revealed not only from attempts to obtain exact equations of momentum and heat flow for a viscous rarefied medium, it is also possible to show that the Euler equations do not agree well with molecular-kinetic concepts of media properties. We shall try to explain this in a concrete example.

Let us consider a rectilinear vortex cord of an infinite length in an ideal medium. In this case the rotor of a velocity of a partial motion beyond the vortex cord will be equal to zero. Consequently, motion of a medium beyond the vortex cord will be potential. Therefore, according to the Euler equation for the steady iso-entropic motion of an ideal continuous medium, we can write such an equality for a velocity field beyond the vortex cord:

$$\frac{W_0^2 r_0^2}{2r^2} + \frac{p_0}{\rho_0^{\gamma}} \frac{\gamma}{(\gamma - 1)} \rho^{\gamma-1} = \text{const.} \qquad (3)$$

Here the subscript "0" denotes the values of quantities taken on the surface of the vortex cord. The equation obtained is valid provided that the medium is continuous. Only in this case are all the mathematical operations which have to be fulfilled while deriving the given equations quite lawful. But if the medium is discrete, there is no reason to consider that these operations are lawful. In this case, following the Klein example, we must use finite difference and derive the equation of motion from purely physical concepts independently of the Euler equations.

(2) In discrete medium we must take as the hydrodynamic element of the volume such a volume as would contain rather a great number of molecules. Only in this case can we expect that the averaged velocities of the regulated motion can change continuously from one point to another.

Let us cut out a ring with a thickness equal to Δl and width Δr in a region beyond the vortex cord. Further, let us cut a curvilinear parallelepiped out of this ring by two planes and let the angle between these planes be equal to $\Delta\theta$. Then, the volume of this parallelepiped can be written in such a form:

$$\Delta\sigma = \bar{r}\Delta\theta\Delta l\Delta r.$$

Here $\bar{r}$ is the average arithmetic value of two radii of the ring. We must choose this hydrodynamic volume thus that it will catch rather a great number of molecules.

Let us designate the average angular velocity of the curvilinear parallelepiped volume as $\bar{\omega}$. Then the centripetal force effecting this hydrodynamic volume will be equal to:

$$\bar{\rho}\,[(r_2^2 - r_1^2)/2]\Delta\theta\Delta l\bar{r}\bar{\omega}^2.$$

where $\bar{\rho}$ is the average density. On the other hand the pressure on the average surface of a curvilinear parallelepiped will be equal to:

$$(p_2 - p_1)\,\bar{r}\Delta\theta\Delta l.$$

Comparing these two forces with each other we shall have:

$$\frac{r_2^2 - r_1^2}{2}\Delta\theta\Delta l\bar{r}\bar{\omega}^2 = -\frac{p_2 - p_1}{\bar{\rho}}\bar{r}\Delta\theta\Delta l. \qquad (4)$$

If the phenomenon occurs in an iso-entropical way, such an equation can be easily obtained from the Poisson equation:

$$p_1 - p_2 = \frac{\gamma}{(\gamma - 1)}\bar{\rho}(\rho_1^{\gamma-1} - \rho_2^{\gamma-1})A$$

$$= \frac{p_0}{\rho_0^{\gamma}}\frac{\gamma}{(\gamma - 1)}\bar{\rho}(\rho_1^{\gamma-1} - \rho_1^{\gamma-1}).$$

Now equation (4) can be transformed in the following way:

$$\frac{r_2^2 - r_1^2}{2}\bar{\omega}^2 = \frac{p_0}{\rho_0^{\gamma}}\frac{\gamma}{(\gamma - 1)}(\rho_1^{\gamma-1} - \rho_2^{\gamma-1}). \qquad (4a)$$

In order to obtain from this equation the relation (3) which follows from the Euler equation it is necessary to exclude $\bar{\omega}^2$. This elimination reserves all the details which result from the necessity of a transition from the discrete medium to the continuous one. Indeed, we suppose that a motion velocity of internal and external points of a ring is determined correspondingly by relations:

$$\frac{W_0 r_0}{r_1} = \omega_1 r_1 = W_{1r}; \quad \frac{W_0 r_0}{r_2} = \omega_2 r_2 = W_{2r}. \qquad (5)$$

Each of these relations is the effect of the potentiality of the flow. If we assume this, then the average angular velocity ω will be equal to:

$$\bar{\omega} = \frac{W_0 r_0}{2}\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right) = \frac{W_0 r_0}{r_1^2 r_2^2}\bar{r}^2.$$

Or substituting the arithmetical mean of $\bar{r} = (r_1 + r_2)/2$ for the geometrical one of $\sqrt{(r_1 r_2)}$ we shall get:

$$\bar{\omega} = \frac{W_0 r_0}{r_1 r_2}.$$

Now we may reduce the equality (4a) to:

$$\frac{W_0^2 r_0^2}{2r_2^2} + \frac{p_0}{\rho_0^{\gamma}}\frac{\gamma}{(\gamma - 1)}\rho_2^{\gamma-1}$$

$$= \frac{W_0^2 r_0^2}{2r_1^2} + \frac{p_0}{\rho_0^{\gamma}}\frac{\gamma}{(\gamma - 1)}\rho_1^{\gamma-1}.$$

As we see, the obtained equality coincides essentially with the equality (3) which was found from the Euler equation.

However, we made an unfounded assumption that external and internal surfaces of a physically small volume have various angular velocities throughout, but for determination of continuously changing average velocities of a regulated motion we must assign a specific physically small volume attributing to it some angular velocity and a corresponding radius-vector. This choice must be accomplished in such a way that the equality will be always valid:

$$\frac{W_0 r_0}{r_i} = \omega_i r_i'.$$

Here r_i determines the location of the physically small volume. It is seen from this equality that we can change arbitrarily the angle velocity ω_i and the radius-vector r_i'. However, from the physical considerations it is clear that the angle velocity of the physically small volume must be constant (ω). Consequently, only the radius-vector r_i' must be chosen in accordance with the physically small volume. It means that the following equalities must be valid for the motion of external and internal surfaces of the physically small volume:

$$\frac{W_0 r_0}{r_1} = \omega r_1'; \qquad \frac{W_0 r_0}{r_2} = \omega r_2'.$$

Hence, it follows:

$$\frac{r_1}{r_2'} = \frac{r_2'}{r_1'} = \beta; \qquad \omega = \frac{W_0 r_0}{r_2 r_1} \beta.$$

If we introduce the value of the angular velocity ω into the equality (4a) we shall obtain:

$$\frac{W_0^2 r_0^2}{2r^2} \beta^2 + \frac{p_0}{\rho_0^\gamma} \frac{\gamma}{(\gamma - 1)} \rho^{\gamma-1} = \text{const.}$$

This equation differs from equation (3). In order to get it with the help of mathematical transformations which require continuity of a medium, we must improve the Euler equations in a suitable manner. It is not difficult to perceive that the improved Euler equation of a motion must have such a form as:

$$\frac{\partial \mathbf{W}}{\partial t} + \beta^2 \operatorname{grad} \frac{W^2}{2} = -\frac{1}{\rho} \operatorname{grad} p. \tag{6}$$

It is convenient to substitute the value β^2 for the following:

$$\beta^2 = 1 - \eta_3.$$

The quantity η_3 is the measure of media continuity with respect to the given hydrodynamic motion. Therefore, we may call this measure a parameter of non-ideal continuity.

By extrapolation of the obtained equation up to the heat velocities it is possible to show that the value of η_3 is equal to:

$$\eta_3 = \frac{2}{3} \frac{L}{D} M.$$

Here, L is the average length of the free path of molecules; D is geometrical sizes of region of flow investigated; M is the Mach local number.

It is impossible to obtain the correction which we have found by solving the Boltzmann gas-kinetic equation, as the possibilities of this equation are restricted, i.e. it solves only the problem of transition of a visible motion into heat.

6

(1) The above-mentioned generalization of the Euler equations allows one to obtain new information on heat transfer phenomena in the flow of a rarefied gas.

As is known, in the problem of gas motion in a cylindrical tube we must take into consideration two equations:

$$\frac{\rho}{2} \frac{\partial W_z^2}{\partial r} + \frac{\partial p}{\partial r} = 0; \qquad \frac{\partial p}{\partial z} = \eta \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial W_z}{\partial r}\right).$$

If we consider the same problem for a flow along a tube of non-ideal continuous media, then proceeding from the Euler generalized equations, we must take into account the following two equations:

$$\rho \frac{(1 - 2\eta_3)}{2} \frac{\partial W_z^2}{\partial r} + \frac{\partial p}{\partial r} = 0;$$

$$\frac{\partial p}{\partial z} = \eta \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial W_z}{\partial r}\right).$$

Near a wall the first equation can be transformed with the help of the equation of state and relations for the temperature jump and slip, i.e. we can use the following formulae:

$$\zeta \frac{\partial W_z}{\partial r} = -W_s; \quad k\frac{\partial T}{\partial r} = \Delta T;$$

$$\frac{\partial p}{\partial r} = R\frac{\mathrm{d}T}{\mathrm{d}r}\rho = \frac{R\Delta T\rho}{k}.$$

As a result of such a transformation, we shall get:

$$\frac{(1-2\eta_3)W_s^2}{\zeta} = \frac{R\Delta T}{k}. \tag{7}$$

On the other hand, proceeding from the law of conservation of energy, we have:

$$\frac{\rho W_s^2}{2} + p = p_0.$$

Hence:

$$W_s^2 = 2R_0\Delta T.$$

Now equation (7) can be rewritten in such a form:

$$2(1-2\eta_3)k = \zeta. \tag{7a}$$

Due to investigations carried out by Gribkova and Shtemenko the relation obtained is of great interest. Indeed, if the relation ζ/k remains constant for all the viscous molecular flows and is equal to 8/15, then from equation (7a) it is possible to calculate a parameter of non-ideal continuity. Its value will be equal to 0·367. On the other hand we know the parameter of non-ideal continuity in an explicit form. Therefore, it can be written:

$$\eta_3 = \frac{2}{3}\frac{L}{D}M = 0{\cdot}367.$$

Hence:

$$\frac{L}{D}M = 0{\cdot}550.$$

For the Mach number equal to unity the size of the flow region where the jump of temperature and flow slip occur will be equal to 1·82L, i.e. about two lengths of the average path of the molecular wandering. Since in this case the Mach number is calculated according to the local value of the velocity of molecules the calculated value of sizes of the region of a flow where the temperature jump and slip occur will be maximum.

Thus, the physically small volume for a molecular-viscous flow must exceed by approximately eight times the volume of a cube whose edge is equal to the average length of the free path of molecules. This result is of great interest from the viewpoint of physics. It shows that it is not obligatory to have a great number of molecules in one cubic centimetre while dealing with continuum consideration of a medium. Consequently, the classical equations which were derived in a phenomenological way are also valid for sufficiently deep rarefication of a gaseous medium.

(2) It is possible to extend the generalization of the Euler equations obtained from the analysis of turbulence in ideal medium to a viscous gas as well. In this case only the equations of motion and energy need corrections; the continuity equation remains unchanged.

If we write the Navier–Stokes equation of motion in symbols:

$$\frac{\mathrm{d}\mathbf{W}}{\mathrm{d}t} = \mathbf{S}$$

then after taking into account the corrections for non-ideal continuity we shall have:

$$\frac{\mathrm{d}\mathbf{W}}{\mathrm{d}t} = S + \eta_3\{\tfrac{1}{2}\operatorname{grad} W^2 + \mathbf{W}\operatorname{div}\mathbf{W} + [\operatorname{rot}\mathbf{W}_1\mathbf{W}]\}.$$

As is known, in aerodynamics of a viscous liquid we write the equation of energy in its general form as follows:

$$\rho c_v\frac{\mathrm{d}T}{\mathrm{d}t} + p\operatorname{div}\mathbf{W} = E_s.$$

Taking into consideration the corrections for non-ideal continuity the equation can be written as follows:

$$\rho c_v\frac{\mathrm{d}T}{\mathrm{d}t} + p\operatorname{div}\mathbf{W} = E_s + \rho\eta_3 W^2\operatorname{div}\mathbf{W} + \rho\eta_3\left(\mathbf{W}, \operatorname{grad}\frac{W^2}{2}\right).$$

The degree of fitness of the equations obtained for the description of the motion of rarefied gases

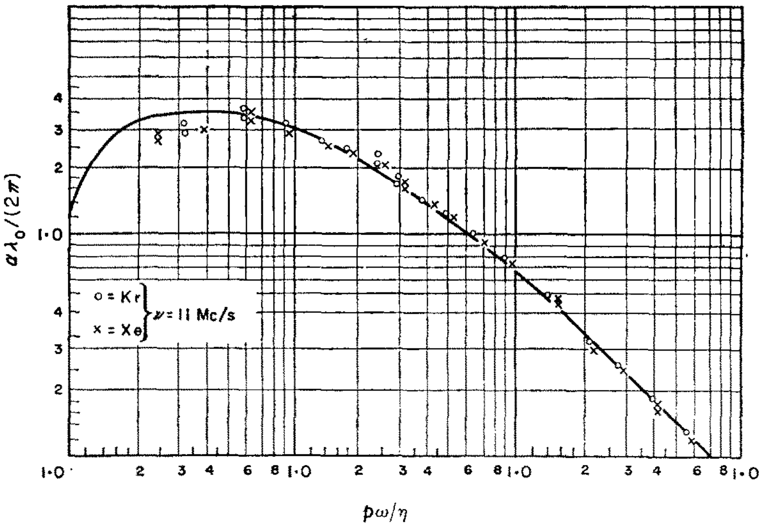

FIG. 1.

Plot of the values $\frac{\alpha}{\beta_0} = \frac{\alpha\,\lambda_0}{2\pi}$ versus $r = \frac{p}{\omega\eta}$

×, ○ —experimental data, ——— —calculated curve according to author's formulae.

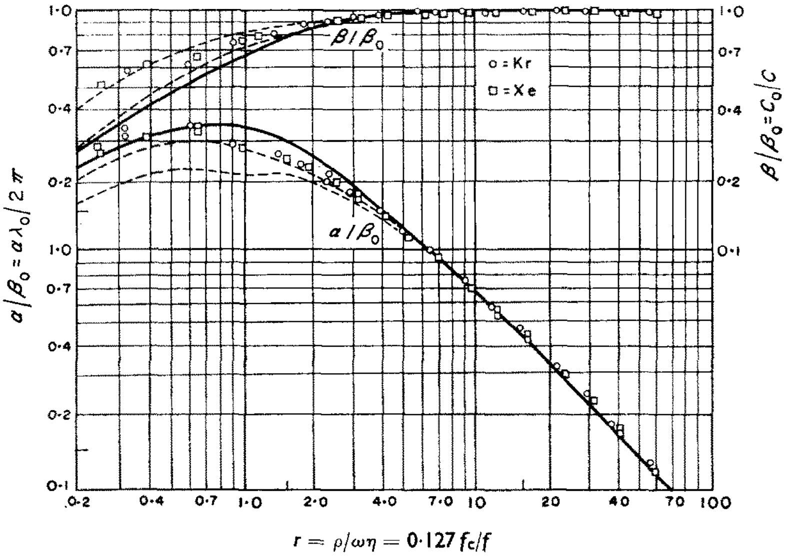

FIG. 2.

Plot of the values $\frac{\alpha}{\beta_0} = \frac{\alpha\,\lambda_0}{2\pi}$ versus $r = \frac{\rho}{\omega\eta}$

○, □ —experimental data, ——— —calculated curve according to Stokes–Kirchoff,

- - - - - —calculated curve according to second approximation Burnett,

— — — —calculated curve according to third approximation Burnett.

can be obtained while examining the dispersion of acoustic waves in an experimental way. It is not difficult to obtain formulae of acoustic dispersion if we use the conditions of compatibility by Hugonio–Hadamard. As is known, the above-mentioned conditions characterize the process of formation and propagation of a front. They allow one to find without any integration (i.e. only by segregation of discontinuity surfaces), the velocity of motion of the latter throughout a space. The peculiarity of any acoustic front results in the fact that when passing through it from the disturbance region to that of rest the first and second derivatives entering the closed system of momentum equations suffer a break of continuity. Not dwelling upon the detailed calculations the author of this paper succeeded in finding the following formulae for the sound absorption coefficient in a rarefied gas:

$$a = \frac{(0{\cdot}7r + 0{\cdot}23)\sqrt{r}}{(r + 0{\cdot}575)^2\sqrt{(r + 0{\cdot}575)}}.$$

The calculations were fulfilled for monatomic gases which were investigated by Greenspan and Thompson. The above-mentioned dimensionless value is designated by r in the given formula.

The Greenspan data on the dispersion of ultrasound waves in xenon and krypton are given in Fig. 1. The continuous curve is plotted according to the theoretical formula.

As is seen, the coincidence of observed and predicted values of the absorption coefficient of acoustic waves may be considered rather good.

It is interesting to compare this result with that of the Navier–Stokes, Burnett and super-Burnett equations.

The Greenspan data obtained from experiments in krypton and xenon and calculations according to theoretical formulae are given in Fig. 2. The continuous curve corresponds to the Stokes–Kirchoff equations. The short dashed curve corresponds to the Burnett equations and the long dashed curve corresponds to the super-Burnett equations.

Hence, the formulae of acoustic dispersion based on the Navier–Stokes equations describe the experiment better than the formulae based on the Burnett and super-Burnett equations.

As for our formula, it coincides completely with the Stokes–Kirchoff formula over the whole range. This formula gives a better agreement with experimental data only at values approximately equal to unity.

Thus, the analysis of this problem carried out in a different manner to the Truesdell analysis leads to just the same conclusion. Undoubtedly, we deal with a crisis in the molecular-kinetic theory when we relate equations of aerodynamics and heat theory for rarefied gases.